Computers and Mathematics with Applications 62 (2011) 3555–3566

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Some new fixed point theorems of generalized nonlinear contractive multivalued maps in complete metric spaces

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ARTICLE INFO

Article history:
Received 12 January 2011
Received in revised form 29 August 2011
Accepted 1 September 2011

Keywords:
τ-function
w-distance
Nonlinear contractive map
Fixed point
Lower semicontinuity
Gauge function

ABSTRACT

In this paper, we present some new fixed point theorems for generalized nonlinear contractive multivalued maps and these results generalize or improve the corresponding recent fixed point results in the literature.

1. Introduction

Let (X, d) be a metric space. Let $\mathcal{C}(X)$ and $\mathcal{CB}(X)$ be the set of all nonempty closed, all nonempty closed bounded subsets of X, respectively. Let $d(x, A)$ denote the distance from x to $A \subseteq X$. Let H denote the Hausdorff metric induced by d, that is,

$$H(A, B) := \max\{\sup_{u \in A} d(u, B), \sup_{v \in B} d(v, A)\}.$$

Following the Banach contraction principle, Nadler [1] and Markin [2] first initiated the study of fixed point theorems for multi-valued contraction maps.

Theorem 1.1 (*Nadler's Fixed Point Theorem [1]*)**.** *Let (X, d) be a complete metric space and let T be a mapping from X into $\mathcal{CB}(X)$. Assume that there exists $\lambda \in [0, 1)$ such that*

$$H(T(x), T(y)) \leq \lambda d(x, y) \quad \textit{for all } x, y \in X.$$

Then there exists $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$.

In 1989, Mizoguchi and Takahashi [3] gave the following result which is a generalization of Theorem 1.1.

Theorem 1.2 (*[3]*)**.** *Let (X, d) be a metric space and let T be a mapping from X into $\mathcal{CB}(X)$. Assume that there exists a map $\varphi : [0, \infty) \to [0, 1)$ such that*

$$\limsup_{r \to t^+} \varphi(r) < 1 \quad \textit{for each } t \in [0, \infty)$$

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0898-1221/$ – see front matter
doi:10.1016/j.camwa.2011.09.001

and

$$H(T(x), T(y)) \leq \varphi(d(x, y)) \cdot d(x, y) \quad \textit{for each } x, y \in X.$$

Then there exists $\bar{x} \in X$ *such that* $\bar{x} \in T(\bar{x})$.

In 2006, Feng and Liu [4] gave the following result.

Theorem 1.3 ([4]). *Let* (X, d) *be a complete metric space, and let* $T : X \to \mathcal{C}(X)$ *be a mapping. If there exist* $b, c \in (0, 1)$, $c < b$, *such that for any* $x \in X$, *there is* $y \in T(x)$ *satisfying the following two conditions:*

(i) $b \cdot d(x, y) \leq d(x, T(x))$; *and*
(ii) $d(y, T(y)) \leq c \cdot d(x, y)$.

Then there exists $\bar{x} \in X$ *such that* $\bar{x} \in T(\bar{x})$ *provided the function* $f(x) := d(x, T(x))$ *is lower semicontinuous.*

In 2007, Klim and Wardowski [5] proved the following result which is a generalization of Theorem 1.3.

Theorem 1.4 ([5]). *Let* (X, d) *be a complete metric space, and let* $T : X \to \mathcal{C}(X)$ *be a mapping. Assume that:*

(i) *the function* $f(x) := d(x, T(x))$ *is lower semicontinuous;*
(ii) *there exist a constant* $b \in (0, 1)$ *and a function* $\varphi : [0, \infty) \to [0, b)$ *such that*

$$\limsup_{r \to t^+} \varphi(r) < b \quad \textit{for each } t \in [0, \infty);$$

and for any $x \in X$, *there is* $y \in T(x)$ *satisfying the following two conditions:*
(a) $b \cdot d(x, y) \leq d(x, T(x))$; *and*
(b) $d(y, T(y)) \leq \varphi(d(x, y)) \cdot d(x, y)$.

Then there exists $\bar{x} \in X$ *such that* $\bar{x} \in T(\bar{x})$.

For these results, there are many generalized results in the literature, for example, see [6–14] and the references therein. Motivated by the above works, in this paper, we present some new fixed point theorems for generalized nonlinear contractive multivalued maps and these results generalize or improve the corresponding recent fixed point results in [5–9,13].

2. Preliminaries

Kada et al. [15] introduced the concept of the w-distance on a metric space as follows:

Definition 2.1 ([15]). Let (X, d) be a metric space. A function $p : X \times X \to [0, \infty)$ is called a w-distance if the following conditions hold:

($w1$) for each $x, y, z \in X$, $p(x, z) \leq p(x, y) + p(y, z)$;
($w2$) for each $x \in X$, $y \to p(x, y)$ is l.s.c.;
($w3$) for each $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

Lin and Du [16] introduced the concept of the τ-function on a metric space as follows:

Definition 2.2 ([16]). Let (X, d) be a metric space. A function $p : X \times X \to [0, \infty)$ is called a τ-function if the following conditions hold:

($\tau1$) for each $x, y, z \in X$, $p(x, z) \leq p(x, y) + p(y, z)$;
($\tau2$) if $x \in X$ and $\{y_n\}_{n\in\mathbb{N}} \subseteq X$ with $\lim_{n\to\infty} y_n = y$ such that $p(x, y_n) \leq M$ for some $M := M(x) > 0$, then $p(x, y) \leq M$;
($\tau3$) for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X with $\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0$, if there exists a sequence $\{y_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} p(x_n, y_n) = 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$;
($\tau4$) for $x, y, z \in X$, $p(x, y) = 0$ and $p(x, z) = 0$ imply $y = z$.

Remark 2.1. It is well known that the metric d is a w-distance and any w-distance is a τ-function, but the converse is not true [16].

Lemma 2.1 ([16]). *Let* (X, d) *be a metric space and* $p : X \times X \to [0, \infty)$ *be any function. Assume that* p *satisfies* ($\tau3$). *If* $\{x_n\}_{n=0}^{\infty}$ *is a sequence in* X *with* $\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0$, *then* $\{x_n\}_{n=0}^{\infty}$ *is a Cauchy sequence in* X.

Lemma 2.2 ([17,18]). *Let* A *be a nonempty closed subset of a metric space* (X, d), *and let* $p : X \times X \to [0, \infty)$ *be a function. Assume that* p *satisfies* ($\tau3$) *and there exists* $u \in X$ *such that* $p(u, u) = 0$. *Then* $p(u, A) = 0$ *if and only if* $u \in A$, *where* $p(u, A) := \inf\{p(u, a) : a \in A\}$.

3. Fixed point theorems (I)

Theorem 3.1. *Let (X, d) be a complete metric space and $p : X \times X \to [0, \infty)$ be a function with properties $(\tau 1)$, $(\tau 2)$, and $(\tau 3)$. Let $T : X \to \mathcal{C}(X)$ be a multivalued map. Let $f : X \to \mathbb{R}$ be defined by $f(x) := p(x, T(x))$ for each $x \in X$. Let $0 < a < 1$, and let $\varphi : [0, \infty) \to [0, 1)$ and $\phi : [0, \infty) \to [a, 1)$ be two functions with the properties:*

$$\limsup_{r \to t^+} \frac{\varphi(r)}{\phi(r)} < 1 \quad \text{for each } t \in [0, \infty). \tag{1}$$

For each $x \in X$, there exists $y \in T(x)$ such that

$$\phi(p(x, T(x))) \cdot p(x, y) \leq p(x, T(x)), \tag{2}$$

and

$$p(y, T(y)) \leq \varphi(p(x, T(x))) \cdot p(x, y). \tag{3}$$

Then there exists $\bar{x} \in X$ such that

(A1) *if T is closed (i.e., $\mathrm{Gr}(T) := \{(x, y) \in X \times X : y \in T(x)\}$ is a closed set), then $\bar{x} \in T(\bar{x})$;*
(A2) *if f is l.s.c., then $p(\bar{x}, T(\bar{x})) = 0$. Furthermore, if $p(\bar{x}, \bar{x}) = 0$, then $\bar{x} \in T(\bar{x})$;*
(A3) *$\inf\{p(x, z) + p(x, T(x)) : x \in X\} > 0$ for each $z \in X$ with $z \notin T(z)$ implies that $\bar{x} \in T(\bar{x})$.*

Proof. Take any point $x_0 \in X$ and let x_0 be fixed. By (2) and (3), there exists $x_1 \in T(x_0)$ such that

$$\phi((p(x_0, T(x_0)))) \cdot p(x_0, x_1) \leq p(x_0, T(x_0)), \tag{4}$$

and

$$p(x_1, T(x_1)) \leq \varphi(p(x_0, T(x_0))) \cdot p(x_0, x_1). \tag{5}$$

Continuing this process, we can choose a sequence $\{x_n\}_{n=1}^{\infty}$ with $x_{n+1} \in T(x_n)$ such that

$$\phi((p(x_n, T(x_n)))) \cdot p(x_n, x_{n+1}) \leq p(x_n, T(x_n)), \tag{6}$$

and

$$p(x_{n+1}, T(x_{n+1})) \leq \varphi(p(x_n, T(x_n))) \cdot p(x_n, x_{n+1}) \tag{7}$$

for each $n \in \mathbb{N} \cup \{0\}$. By (6) and (7), for each $n \in \mathbb{N} \cup \{0\}$, we have:

$$p(x_{n+1}, T(x_{n+1})) < \frac{\varphi(p(x_n, T(x_n))) \cdot p(x_n, T(x_n))}{\phi(p(x_n, T(x_n)))}. \tag{8}$$

Clearly, $\{p(x_n, T(x_n))\}_{n=0}^{\infty}$ is a nonincreasing sequence in $[0, \infty)$. Then there exists $\delta \geq 0$ such that

$$\delta = \lim_{n \to \infty} p(x_n, T(x_n)) = \inf\{p(x_n, T(x_n)) : n \in \mathbb{N} \cup \{0\}\}. \tag{9}$$

Suppose that $\delta > 0$. By (1), (8) and (9),

$$\delta \leq \lim_{p(x_n, T(x_n)) \to \delta^+} \frac{\varphi(p(x_n, T(x_n))) \cdot p(x_n, T(x_n))}{\phi(p(x_n, T(x_n)))} < \delta.$$

And this is a contradiction. Thus $\delta = 0$. By (1), (9), and $\delta = 0$, there exist $n_0 \in \mathbb{N}$ and $0 < q < 1$ such that

$$\frac{\varphi(p(x_n, T(x_n)))}{\phi(p(x_n, T(x_n)))} < q \tag{10}$$

for all $n \geq n_0$. For $n \in \mathbb{N}$ with $n > n_0$, by (6)–(8) and (10), we get:

$$\begin{aligned} p(x_n, x_{n+1}) &\leq \frac{p(x_n, T(x_n))}{\phi(p(x_n, T(x_n)))} \\ &\leq \frac{\varphi(p(x_{n-1}, T(x_{n-1}))) \cdot p(x_{n-1}, x_n)}{\phi(p(x_n, T(x_n)))} \\ &\leq \frac{\varphi(p(x_{n-1}, T(x_{n-1})))}{\phi(p(x_n, T(x_n)))} \cdot \frac{p(x_{n-1}, T(x_{n-1}))}{\phi(p(x_{n-1}, T(x_{n-1})))} \\ &\leq \frac{q \cdot p(x_{n-1}, T(x_{n-1}))}{a} \end{aligned}$$

$$\le \frac{q^2 \cdot p(x_{n-2}, T(x_{n-2}))}{a}$$
$$\vdots$$
$$\le \frac{q^{n-n_0} \cdot p(x_{n_0}, T(x_{n_0}))}{a}.$$

Hence, for $m, n \in \mathbb{N}$ with $m > n > n_0$,

$$p(x_n, x_m) \le \sum_{i=n}^{m-1} p(x_i, x_{i+1}) \le \frac{q^{n-n_0} p(x_{n_0}, T(x_{n_0}))}{a(1-q)}. \tag{11}$$

Since $0 < q < 1$, we know that

$$\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0. \tag{12}$$

By Lemma 2.1, $(\tau 3)$, and (12), we know that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in X. Since X is complete, there exists $\bar{x} \in X$ such that $x_n \to \bar{x}$ as $n \to \infty$. By $(\tau 2)$,

$$p(x_n, \bar{x}) \le \frac{q^{n-n_0} p(x_{n_0}, T(x_{n_0}))}{a(1-q)}$$

for all $n > n_0$. For each $n > n_0$, let

$$a_n := \frac{q^{n-n_0} p(x_{n_0}, T(x_{n_0}))}{a(1-q)}.$$

If (A1) holds, then T is closed. Since $x_{n+1} \in T(x_n)$ for each $n \in \mathbb{N} \cup \{0\}$ and $x_n \to \bar{x}$ as $n \to \infty$, $\bar{x} \in T(\bar{x})$. If (A2) holds, then we have:

$$0 \le p(\bar{x}, T(\bar{x})) = f(\bar{x}) \le \liminf_{n\to\infty} p(x_n, T(x_n)) = 0.$$

Then $p(\bar{x}, T(\bar{x})) = 0$. Furthermore, if $p(\bar{x}, \bar{x}) = 0$, then $\bar{x} \in T(\bar{x})$.

If (A3) holds, suppose that $\bar{x} \notin T(\bar{x})$, then we have:

$$\begin{aligned} 0 &< \inf\{p(x, \bar{x}) + p(x, T(x)) : x \in X\} \\ &\le \inf\{p(x_n, \bar{x}) + p(x_n, T(x_n)) : n \in \mathbb{N},\ \text{and } n > n_0\} \\ &\le \inf\{p(x_n, \bar{x}) + p(x_n, x_{n+1}) : n \in \mathbb{N} \text{ and } n > n_0\} \\ &\le \inf\{2a_n : n \in \mathbb{N} \text{ and } n > n_0\} = 0. \end{aligned}$$

And this is a contradiction. Hence, $\bar{x} \in T(\bar{x})$. □

Remark 3.1.

(i) Theorem 3.1 generalizes Theorem 2.2 in [14] since p is not a metric and we do not assume that $\varphi(t) < \phi(t)$ for each $t \in [0, \infty)$.

(ii) Theorem 3.1 is different from Theorem 2.1 in [10]. In Theorem 3.1, p is not a metric and we do not assume that $\liminf_{r\to 0^+} \phi(r) > 0$. But, for Theorem 2.1 in [10], Liu et al. assume that $\limsup_{r\to t^+} \frac{\varphi(r)}{\phi(r)} < 1$ for each $t \in [0, \sup f(X))$, where $f(x) := p(x, T(x))$.

Corollary 3.1. *Let (X, d) be a complete metric space and let $p : X \times X \to [0, \infty)$ be a function with properties $(\tau 1)$, $(\tau 2)$, and $(\tau 3)$. Let $T : X \to \mathcal{C}(X)$ be a multivalued map. Let $f : X \to \mathbb{R}$ be defined by $f(x) := p(x, T(x))$ for each $x \in X$. Let $\varphi : [0, \infty) \to [a, 1)$, $0 < a < 1$, be a function with the properties:*

$$\lim_{r\to t^+} \sup \varphi(r) < 1 \quad \text{for each } t \in [0, \infty). \tag{13}$$

For each $x \in X$, there exists $y \in T(x)$ such that

$$\sqrt{\varphi(p(x, T(x)))} \cdot p(x, y) \le p(x, T(x)), \tag{14}$$

and

$$p(y, T(y)) \le \varphi(p(x, T(x))) \cdot p(x, y). \tag{15}$$

Then there exists $\bar{x} \in X$ such that

(A1) *if T is closed, then $\bar{x} \in T(\bar{x})$;*
(A2) *if f is l.s.c., then $p(\bar{x}, T(\bar{x})) = 0$. Furthermore, if $p(\bar{x}, \bar{x}) = 0$, then $\bar{x} \in T(\bar{x})$;*
(A3) *$\inf\{p(x, z) + p(x, T(x)) : x \in X\} > 0$ for each $z \in X$ with $z \notin T(z)$ implies that $\bar{x} \in T(\bar{x})$.*

Proof. Let $\phi : [0, \infty) \to [a, 1)$ be defined by $\phi(t) = \sqrt{\varphi(t)}$ for each $t \in [0, \infty)$. By Theorem 3.1, we get the proof of Corollary 3.1. □

Remark 3.2. If in Corollary 3.1, if p is a w-distance, then (A2) (resp., (A3)) is reduced to Theorem 2.1 (resp., Theorem 2.2) in [13]. Furthermore, if p is a metric, then Corollary 3.1 (A2) is reduced to Theorem 2.1 in [7].

Theorem 3.2. *Let (X, d) be a complete metric space and $p : X \times X \to [0, \infty)$ be a function with properties $(\tau 1)$, $(\tau 2)$, and $(\tau 3)$. Let $T : X \to \mathcal{C}(X)$ be a multivalued map. Let $f : X \to \mathbb{R}$ be defined by $f(x) := p(x, T(x))$ for each $x \in X$. Let $\varphi : [0, \infty) \to [0, 1)$ and $\phi : [0, \infty) \to [a, 1)$, $0 < a < 1$, be two functions with the properties:*

$$\varphi(t) \leq (\phi(t))^2 \quad \text{for each } t \in [0, \infty), \tag{16}$$

and

$$\limsup_{r \to t^+} \phi(r) < 1 \quad \text{for each } t \in [0, \infty). \tag{17}$$

For each $x \in X$, there exists $y \in T(x)$ such that

$$\phi(p(x, y)) \cdot p(x, y) \leq p(x, T(x)), \tag{18}$$

and

$$p(y, T(y)) \leq \varphi(p(x, y)) \cdot p(x, y). \tag{19}$$

Then there exists $\bar{x} \in X$ such that

(A1) *if T is closed, then $\bar{x} \in T(\bar{x})$;*
(A2) *if f is l.s.c., then $p(\bar{x}, T(\bar{x})) = 0$. Furthermore, if $p(\bar{x}, \bar{x}) = 0$, then $\bar{x} \in T(\bar{x})$;*
(A3) *$\inf\{p(x, z) + p(x, T(x)) : x \in X\} > 0$ for each $z \in X$ with $z \notin T(z)$ implies that $\bar{x} \in T(\bar{x})$.*

Proof. Following the lines of Theorem 3.1, we can construct a sequence $\{x_n\}_{n=0}^{\infty}$ in X such that $x_{n+1} \in T(x_n)$, and

$$\phi(p(x_n, x_{n+1})) \cdot p(x_n, x_{n+1}) \leq p(x_n, T(x_n)) \tag{20}$$

and

$$p(x_{n+1}, T(x_{n+1})) \leq \varphi(p(x_n, x_{n+1})) \cdot p(x_n, x_{n+1}) \tag{21}$$

for all $n \in \mathbb{N} \cup \{0\}$. For each $n \in \mathbb{N} \cup \{0\}$, by (20) and (21), we get

$$p(x_{n+1}, T(x_{n+1})) \leq \frac{\varphi(p(x_n, x_{n+1}))}{\phi(p(x_n, x_{n+1}))} \cdot p(x_n, T(x_n)). \tag{22}$$

Then $\{p(x_n, T(x_n))\}_{n=0}^{\infty}$ is a nonincreasing sequence. Hence, there exists $\delta \geq 0$ such that

$$\delta = \lim_{n \to \infty} p(x_n, T(x_n)) = \inf\{p(x_n, T(x_n)) : n \in \mathbb{N} \cup \{0\}\}. \tag{23}$$

Besides, $\{p(x_n, x_{n+1})\}_{n=0}^{\infty}$ is a bounded sequence. Then there exists $\theta \geq 0$ such that

$$\liminf_{n \to \infty} p(x_n, x_{n+1}) = \theta. \tag{24}$$

Clearly, $\delta \leq \theta$. Next, we want to show that $\theta \leq \delta$ and this implies that $\theta = \delta$.

Case 1: If $\delta = 0$, then

$$0 \leq \theta = \liminf_{n \to \infty} p(x_n, x_{n+1}) \leq \liminf_{n \to \infty} \frac{p(x_n, T(x_n))}{a} = 0.$$

Case 2: If $\delta > 0$.

Suppose that $\theta > \delta$. By (23) and (24), there exists $n_0 \in \mathbb{N}$ such that for each $n > n_0$,

$$p(x_n, T(x_n)) < \delta + \frac{\theta - \delta}{3}, \tag{25}$$

and

$$\theta - \frac{\theta - \delta}{3} < p(x_n, x_{n+1}). \tag{26}$$

By (20), (25) and (26), for $n > n_0$,

$$\phi(p(x_n, x_{n+1})) \le h, \quad \text{where } h := 1 - \frac{\theta - \delta}{2\theta + \delta}. \tag{27}$$

By (16) and (27), for $n > n_0$,

$$\frac{\varphi(p(x_n, x_{n+1}))}{\phi(p(x_n, x_{n+1}))} \le \phi(p(x_n, x_{n+1})) \le h. \tag{28}$$

And this implies that for each $k \in \mathbb{N}$,

$$p(x_{n_0+k}, T(x_{n_0+k})) \le h^k \cdot p(x_{n_0}, T(x_{n_0})). \tag{29}$$

Since $0 < \delta$ and $0 < h < 1$, there exists $k_0 \in \mathbb{N}$ such that

$$h^k \cdot p(x_{n_0}, T(x_{n_0})) < \delta. \tag{30}$$

By (23), (29) and (30), we get

$$\delta \le p(x_{n_0+k_0}, T(x_{n_0+k_0})) \le h^{k_0} \cdot p(x_{n_0}, T(x_{n_0})) < \delta. \tag{31}$$

And this is a contradiction. So, $\theta \le \delta$ and this implies that $\theta = \delta$.

Next, we want to show that $\theta = 0$. Since $\theta = \delta \le p(x_n, T(x_n)) \le p(x_n, x_{n+1})$ and $\liminf_{n\to\infty} p(x_n, x_{n+1}) = \theta$, we can read as $\liminf_{n\to\infty} p(x_n, x_{n+1}) = \theta^+$, and there exists a subsequence $\{p(x_{n_k}, x_{n_{k+1}})\}_{k=0}^{\infty}$ of $\{p(x_n, x_{n+1})\}_{n=0}^{\infty}$ such that

$$\lim_{k\to\infty} p(x_{n_k}, x_{n_{k+1}}) = \theta^+. \tag{32}$$

By (16), (17), and (32),

$$\limsup_{p(x_{n_k}, x_{n_{k+1}})\to\theta^+} \frac{\varphi(p(x_{n_k}, x_{n_{k+1}}))}{\phi(p(x_{n_k}, x_{n_{k+1}}))} \le \limsup_{p(x_{n_k}, x_{n_{k+1}})\to\theta^+} \phi(p(x_{n_k}, x_{n_{k+1}})) < 1. \tag{33}$$

Hence,

$$\delta = \limsup_{k\to\infty} p(x_{n_{k+1}}, T(x_{n_{k+1}})) \le \limsup_{p(x_{n_k}, x_{n_{k+1}})\to\theta^+} \phi(p(x_{n_k}, x_{n_{k+1}})) \cdot \delta.$$

And this implies that $\delta = \theta = 0$.

By (17), (23), and $\delta = 0$, there exists $n_0 \in \mathbb{N}$ and $q \in (0, 1)$ such that for each $n \ge n_0$,

$$\phi(p(x_n, T(x_n))) < q. \tag{34}$$

Then for each $n \ge n_0$, we have:

$$\begin{aligned}
p(x_n, x_{n+1}) &\le \frac{p(x_n, T(x_n))}{\phi(p(x_n, x_{n+1}))} \\
&\le \frac{\varphi(p(x_{n-1}, x_n))}{\phi(p(x_n, x_{n+1}))} \cdot p(x_{n-1}, x_n) \\
&\le \frac{\varphi(p(x_{n-1}, x_n))}{\phi(p(x_n, x_{n+1}))} \cdot \frac{p(x_{n-1}, T(x_{n-1}))}{\phi(p(x_{n-1}, x_n))} \\
&\le \frac{q \cdot p(x_{n-1}, T(x_{n-1}))}{a} \\
&\le \frac{q^2 \cdot p(x_{n-2}, T(x_{n-2}))}{a} \\
&\vdots \\
&\le \frac{q^{n-n_0} \cdot p(x_{n_0}, T(x_{n_0}))}{a}.
\end{aligned}$$

Hence, for each $m, n \in \mathbb{N}$ with $m > n \ge n_0$,

$$p(x_n, x_m) \le \sum_{i=n}^{m-1} p(x_i, x_{i+1}) \le \frac{q^{n-n_0} \cdot p(x_{n_0}, T(x_{n_0}))}{a(1-q)}. \tag{35}$$

Since $q \in (0, 1)$,

$$\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0. \tag{36}$$

And this implies that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in X. Proceeding as in the proof of Theorem 3.1, we can complete the proof of Theorem 3.2. □

Remark 3.3.

(i) Theorem 3.2 is different from Theorems 2.2 and 2.4 in [8]. Indeed, p is not a w-distance, and we assume that $\varphi(t) \le (\phi(t))^2$ for each $t \in [0, \infty)$.

(ii) Theorem 3.2 is different from Theorem 2.1 in [5]. Indeed, p is not a metric and we assume that $\varphi(t) \le (\phi(t))^2$ for each $t \in [0, \infty)$. But, for Theorem 2.1 in [5], Klim et al. assume that $\varphi : [0, \infty) \to [0, b), 0 < b < 1$, and $\limsup_{r\to t^+} \varphi(r) < b$.

(iii) Theorem 3.2 is different from Theorem 6 in [6] and Theorem 2.3 in [9]. Indeed, in Theorem 3.2, p is not a metric and not a w-distance. But, for Theorem 6 in [6] and Theorem 2.3 in [9], $\varphi(t) < \phi(t)$ and $\limsup_{r\to t^+} \varphi(r) < \limsup_{r\to t^+} \phi(r)$ for each $t \in [0, \infty)$.

(iv) Theorem 3.2 is different from Theorem 2.3 in [10]. In Theorem 3.2, p is not a metric and we do not assume that $\liminf_{r\to 0^+} \phi(r) > 0$. But, for Theorem 2.3 in [10], Liu et al. assume that $\limsup_{r\to t^+} \frac{\varphi(r)}{\phi(r)} < 1$ for each $t \in [0, \mathrm{diam}(X))$.

(v) Theorem 3.2 is different from Theorem 2.4 in [14]. Indeed, p is not a metric in Theorem 3.2. But for Theorem 2.4 in [14], Nicolae gave the following generalized conditions: $\varphi(t) < \phi(t)$ and $\limsup_{r\to t^+} \frac{\varphi(t)}{\phi(t)} < 1$ for each $t \in [0, \infty)$.

Corollary 3.2. *Let (X, d) be a complete metric space and $p : X \times X \to [0, \infty)$ be a function with properties $(\tau 1)$, $(\tau 2)$, and $(\tau 3)$. Let $T : X \to \mathcal{C}(X)$ be a multivalued map. Let $f : X \to \mathbb{R}$ be defined by $f(x) := p(x, T(x))$ for each $x \in X$. Let $\varphi : [0, \infty) \to [a, 1), 0 < a < 1$, be a function with the properties:*

$$\lim_{r\to t^+} \sup \varphi(r) < 1 \quad \textit{for each } t \in [0, \infty).$$

For each $x \in X$, there exists $y \in T(x)$ such that

$$\sqrt{\varphi(p(x, y))}p(x, y) \le p(x, T(x))$$

and

$$p(y, T(y)) \le \varphi(p(x, y))p(x, y).$$

Then there exists $\bar{x} \in X$ such that

(A1) *if T is closed, then $\bar{x} \in T(\bar{x})$;*
(A2) *if f is l.s.c., then $p(\bar{x}, T(\bar{x})) = 0$. Furthermore, if $p(\bar{x}, \bar{x}) = 0$, then $\bar{x} \in T(\bar{x})$;*
(A3) *$\inf\{p(x, z) + p(x, T(x)) : x \in X\} > 0$ for each $z \in X$ with $z \notin T(z)$ implies that $\bar{x} \in T(\bar{x})$.*

Proof. Let $\phi : [0, \infty) \to [a, 1)$ be defined by $\phi(t) = \sqrt{\varphi(t)}$ for each $t \in [0, \infty)$. By Theorem 3.2, we get the proof of Corollary 3.2. □

Remark 3.4. In Corollary 3.2, if p is a w-distance, then (A2) (resp., (A3)) is reduced to Theorem 2.3 (resp., Theorem 2.5) in [13]. Furthermore, if p is a metric, then Corollary 3.2 (A2) is reduced to Theorem 2.2 in [7].

Theorem 3.3. *Let (X, d) be a complete metric space and $p : X \times X \to [0, \infty)$ be a function with properties $(\tau 1)$, $(\tau 2)$, and $(\tau 3)$. Let $T : X \to \mathcal{C}(X)$ be a multivalued map. Let $f : X \to \mathbb{R}$ be defined by $f(x) := p(x, T(x))$ for each $x \in X$. Let $\varphi : [0, \infty) \to [0, 1)$ be a function with the properties:*

$$\limsup_{r\to t^+} \varphi(r) < 1 \quad \textit{for each } t \in [0, \infty). \tag{37}$$

For each $x \in X$, there exists $y \in T(x)$ such that

$$p(x, y) \le (2 - \varphi(p(x, y))) \cdot p(x, T(x)) \tag{38}$$

and

$$p(y, T(y)) \le \varphi(p(x, y)) \cdot p(x, y). \tag{39}$$

Then there exists $\bar{x} \in X$ such that

(A1) *if T is closed, then $\bar{x} \in T(\bar{x})$;*
(A2) *if f is l.s.c., then $p(\bar{x}, T(\bar{x})) = 0$. Furthermore, if $p(\bar{x}, \bar{x}) = 0$, then $\bar{x} \in T(\bar{x})$;*
(A3) *$\inf\{p(x, z) + p(x, T(x)) : x \in X\} > 0$ for each $z \in X$ with $z \notin T(z)$ implies that $\bar{x} \in T(\bar{x})$.*

Proof. Following the lines of Theorem 3.1, we can construct a sequence $\{x_n\}_{n=0}^{\infty}$ in X such that $x_{n+1} \in T(x_n)$, and

$$p(x_n, x_{n+1}) \leq (2 - \varphi(p(x_n, x_{n+1}))) \cdot p(x_n, T(x_n)), \tag{40}$$

and

$$p(x_{n+1}, T(x_{n+1})) \leq \varphi(p(x_n, x_{n+1})) \cdot p(x_n, x_{n+1}) \tag{41}$$

for all $n \in \mathbb{N} \cup \{0\}$. Let $\eta : [0, \infty) \to [0, 1)$ be defined by $\eta(t) := \varphi(t) \cdot (2 - \varphi(t))$ for each $t \in [0, \infty)$. Clearly, $0 \leq \eta(t) < 1$ for each $t \in [0, \infty)$. By (37),

$$\limsup_{r \to t^+} \eta(r) < 1 \quad \text{for each } t \in [0, \infty). \tag{42}$$

By (40) and (41),

$$p(x_{n+1}, T(x_{n+1})) \leq \eta(p(x_n, x_{n+1})) \cdot p(x_n, T(x_n)). \tag{43}$$

By (43), $\{p(x_n, T(x_n))\}_{n=0}^{\infty}$ is strictly decreasing. Therefore, there exists $\delta \geq 0$ such that

$$\delta = \lim_{n \to \infty} p(x_n, T(x_n)) = \inf\{p(x_n, T(x_n)) : n \in \mathbb{N}\}. \tag{44}$$

Besides, $\{p(x_n, x_{n+1})\}_{n=0}^{\infty}$ is a bounded sequence. Then there exists $\theta \geq 0$ such that

$$\liminf_{n \to \infty} p(x_n, x_{n+1}) = \theta. \tag{45}$$

Clearly, $\delta \leq \theta$. Now, we want to show that $\theta \leq \delta$ and this implies that $\delta = \theta$.
Case 1: If $\delta = 0$, then

$$0 \leq \theta \leq 2 \liminf_{n \to \infty} 2p(x_n, T(x_n)) = 0.$$

Case 2: If $\delta > 0$, then suppose that $\theta > \delta$. By (44) and (45), there exists $n_0 \in \mathbb{N}$ such that

$$p(x_n, T(x_n)) < \delta + \frac{\theta - \delta}{4} \tag{46}$$

and

$$\theta - \frac{\theta - \delta}{4} < p(x_n, x_{n+1}) \tag{47}$$

for each $n > n_0$. By (46) and (47), for each $n > n_0$, we get

$$\eta(p(x_n, x_{n+1})) < h, \quad \text{where } h := 1 - \left[\frac{2(\theta - \delta)}{3\delta + \theta}\right]^2. \tag{48}$$

Clearly, $h < 1$. By (43), for each $k \in \mathbb{N}$,

$$p(x_{n_0+k_0}, T(x_{n_0+k_0})) \leq h^k \cdot p(x_{n_0}, T(x_{n_0})). \tag{49}$$

Since $\delta > 0$ and $h < 1$, there exists $k_0 \in \mathbb{N}$ such that

$$\delta \leq p(x_{n_0+k}, T(x_{n_0+k})) \leq h^{k_0} \cdot p(x_{n_0}, T(x_{n_0})) < \delta. \tag{50}$$

And this is a contradiction. So, $\theta \leq \delta$ and this implies that $\theta = \delta$.

Next, we want to show that $\theta = 0$. Since $\theta = \delta \leq p(x_n, T(x_n)) \leq p(x_n, x_{n+1})$ and $\liminf_{n\to\infty} p(x_n, x_{n+1}) = \theta$, we can read as $\liminf_{n\to\infty} p(x_n, x_{n+1}) = \theta^+$, and there exists a subsequence $\{p(x_{n_k}, x_{n_k+1})\}_{k=0}^{\infty}$ of $\{p(x_n, x_{n+1})\}_{n=0}^{\infty}$ such that

$$\lim_{n \to \infty} p(x_{n_k}, x_{n_k+1}) = \theta^+. \tag{51}$$

By (37) and (51),

$$\limsup_{p(x_{n_k}, x_{n_k+1}) \to \theta^+} \varphi(p(x_{n_k}, x_{n_k+1})) < 1. \tag{52}$$

By (41) and (52),

$$\delta \leq \limsup_{k \to \infty} p(x_{n_k}, T(x_{n_k})) \leq \limsup_{p(x_{n_k}, x_{n_k+1}) \to \theta^+} \varphi(p(x_{n_k}, x_{n_k+1})) \cdot \delta. \tag{53}$$

And this implies that $\delta = \theta = 0$. By (42) and $\delta = 0$, there exists $n_0 \in \mathbb{N}$ and $q \in (0, 1)$ such that for each $n \geq n_0$,

$$\eta(p(x_n, x_{n+1})) < q. \tag{54}$$

Then for each $n > n_0$, we have:

$$\begin{aligned} p(x_n, x_{n+1}) &\leq (2 - \varphi(p(x_n, x_{n+1}))) \cdot p(x_n, T(x_n)) \\ &\leq (2 - \varphi(p(x_n, x_{n+1}))) \cdot \eta(p(x_{n-1}, x_n)) \cdot p(x_{n-1}, T(x_{n-1})) \\ &\leq 2q \cdot p(x_{n-1}, T(x_{n-1})) \\ &\leq 2q^{n-n_0} \cdot p(x_{n_0}, T(x_{n_0})). \end{aligned}$$

Hence, for each $m, n \in \mathbb{N}$ with $m > n > n_0$, we have:

$$p(x_n, x_m) \leq \sum_{i=n}^{m-1} p(x_i, x_{i+1}) \leq \frac{2 \cdot q^{n-n_0} \cdot p(x_{n_0}, T(x_{n_0}))}{1-q}. \tag{55}$$

And this implies that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in X. Proceeding as in the proof of Theorem 3.1, we can complete the proof of Theorem 3.3. □

Remark 3.5. In Theorem 3.3, if p is a w-distance, then (A2) (resp., (A3)) is reduced to Theorem 2.1 (resp., Theorem 2.2) in [9]. Furthermore, if p is a metric, then Theorem 3.3 (A2) is reduced to Theorem 5 in [6].

The following is a simple example for Theorems 3.1–3.3.

Example 3.1. Let $X = [0, 1]$ be a metric space with the usual metric d. Let $p : X \times X \to [0, \infty)$ be defined by $p(x, y) := x$ for each $(x, y) \in X \times X$. Then p satisfies properties $(\tau 1)$, $(\tau 2)$, and $(\tau 3)$. Note that p is not a w-distance. Let $T : X \to \mathcal{C}(X)$ be defined by $T(x) := \left\{\frac{x^2}{4}\right\}$ for each $x \in X$. Let $\varphi : [0, \infty) \to [0, 1)$ and $\phi : [0, \infty) \to \left[\frac{1}{4}, 1\right)$ be defined by $\varphi(t) := \frac{1}{4}$ and $\phi(t) := \frac{1}{2}$ for each $t \in [0, \infty)$. By Theorem 3.1 or Theorem 3.2 or Theorem 3.3, there exists $\bar{x} \in X$ such that $p(\bar{x}, T(\bar{x})) = 0$. Indeed, $\bar{x} = 0 \in T(\bar{x})$. □

Example 3.2. Let $X := [0, 1]$ be a metric space with the usual metric d. Let $p : X \times X \to [0, \infty)$ be defined by $p(x, y) := x$ for each $(x, y) \in X \times X$. Then p satisfies properties $(\tau 1)$, $(\tau 2)$, and $(\tau 3)$, but p is not a w-distance. Let $T : X \to \mathcal{C}(X)$ be defined as Example 3.1 in [5]:

$$T(x) := \begin{cases} \left\{\dfrac{x^2}{2}\right\} & \text{if } x \in \left[0, \dfrac{15}{32}\right) \cup \left(\dfrac{15}{32}, 1\right] \\ \left\{\dfrac{17}{96}, \dfrac{1}{4}\right\} & \text{if } x = \dfrac{15}{32}. \end{cases}$$

Let $\varphi : X \to [0, 1)$ be defined as Example 1 in [7]:

$$\varphi(t) := \begin{cases} \max\left\{\dfrac{3}{4}, \dfrac{23t}{12}\right\} & \text{if } t \in \left[0, \dfrac{1}{2}\right] \\ \left\{\dfrac{23}{24}\right\} & \text{if } t \in \left(\dfrac{1}{2}, \infty\right) \end{cases}$$

and let $\phi : [0, \infty) \to \left[\frac{1}{12}, 1\right)$ be defined by $\phi(t) := \sqrt{\varphi(t)}$. Clearly, $f(x) = p(x, T(x)) = x$ is a continuous function. Clearly, $\limsup_{r \to t^+} \varphi(r) < 1$ for each $t \in [0, \infty)$.

If $x \in \left[0, \frac{15}{32}\right) \cup \left(\frac{15}{32}, 1\right]$, then $y = \frac{x^2}{2}$, and

$$\phi(p(x, y)) \cdot p(x, y) = \phi(x) \cdot x < x = p(x, T(x))$$

$$p(y, T(y)) := \frac{x^2}{2} < \varphi(x) \cdot x = \varphi(p(x, y)) \cdot p(x, y).$$

If $x = \frac{15}{32}$, then for $y = \frac{1}{4}$ $\left(\text{resp., } y = \frac{17}{96}\right)$

$$\phi(p(x, y)) \cdot p(x, y) = \phi(x) \cdot x < x = p(x, T(x))$$

$$p(y, T(y)) := y < \frac{105}{256} = \varphi(x) \cdot x = \varphi(p(x, y)) \cdot p(x, y).$$

By Theorem 3.2, there exists $\bar{x} \in X$ such that $p(\bar{x}, T(\bar{x})) = 0$. Indeed, $\bar{x} = 0 \in T(\bar{x})$. □

4. Fixed point theorems (II)

In this section, we present another generalization of Nadler's fixed point theorem. Throughout this section, let J denote an interval on $[0, \infty)$ containing 0, that is an interval of the form $[0, A]$, $[0, A)$, or $[0, \infty)$, and let $\overline{B}(x, r)$ denote the closed ball centred at x with radius $r > 0$.

Definition 4.1 (*[19]*)**.** A nondecreasing function $\psi : J \to J$ is said to be a Bianchini–Grandolfi gauge function if

$$\sigma(t) := \sum_{n=0}^{\infty} \psi^n(t) < \infty, \quad \text{for all } t \in J.$$

Note that $\sigma(t) := \sigma(\psi(t)) + t$. Clearly, $\sigma(t) \geq t$ for each $t \in J$.

Theorem 4.1. *Let (X, d) be a complete metric space, D be a closed subset of X, $\psi : J \to J$ be a Bianchini–Grandolfi gauge function and $T : D \to \mathcal{C}(X)$ such that $T(x) \cap D \neq \emptyset$ and*

$$d(y, T(y) \cap D) \leq \psi(d(x, y)) \tag{56}$$

for all $x \in D$, $y \in T(x) \cap D$ with $d(x, y) \in J$. Moreover, the strict inequality holds when $d(x, y) \neq 0$. Suppose $x_0 \in D$ is such that $d(x_0, z) \in J$ for some $z \in T(x_0) \cap D$. Then there exists $\bar{x} \in D$ such that $\bar{x} \in T(\bar{x})$ if one of the following conditions hold:

(A1) *T is closed;*
(A2) *$f(x) := d(x, T(x))$ is l.s.c.;*
(A3) *$\inf\{d(x, z) + d(x, T(x) \cap D) : x \in D\} > 0$ for each $z \in D$ with $z \notin T(z)$.*

Proof. Take $x_1 = z \in T(x_0) \cap D$. If $d(x_0, x_1) = 0$, then $x_0 = x_1$ and it is a fixed point of T. If $d(x_0, x_1) \neq 0$, then we define $\rho_0 := \sigma(d(x_0, x_1))$, where σ is defined as in Definition 4.1. Clearly, $d(x_0, x_1) \leq \rho_0$, $x_1 \in \overline{B}(x_0, \rho_0)$, and

$$d(x_1, T(x_1) \cap D) < \psi(d(x_0, x_1)). \tag{57}$$

Then there exists $x_2 \in T(x_1) \cap D$ such that $d(x_1, x_2) \leq \psi(d(x_0, x_1))$. If $d(x_1, x_2) = 0$, then $x_1 = x_2$ and it is a fixed point of T. If $d(x_1, x_2) \neq 0$, then $d(x_1, x_2) \in J$,

$$d(x_0, x_2) \leq d(x_0, x_1) + d(x_1, x_2) \leq d(x_0, x_1) + \psi(d(x_0, x_1)) \leq \rho_0, \tag{58}$$

and

$$d(x_2, T(x_2) \cap D) < \psi(d(x_1, x_2)). \tag{59}$$

Then $x_2 \in \overline{B}(x_0, \rho_0)$, and there exists $x_3 \in T(x_2) \cap D$ such that $d(x_2, x_3) \leq \psi(d(x_1, x_2))$. Clearly, $d(x_2, x_3) \in J$. Furthermore, we know that

$$\begin{aligned} d(x_0, x_3) &\leq d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) \\ &\leq d(x_0, x_1) + \psi(d(x_0, x_1)) + \psi^2(d(x_0, x_1)) \\ &\leq \sum_{n=0}^{\infty} \psi^n(d(x_0, x_1)) = \rho_0. \end{aligned}$$

Then $x_3 \in \overline{B}(x_0, \rho_0)$. Continuing this process, we obtain a sequence $\{x_n\}_{n=0}^{\infty}$ such that for each $n \in \mathbb{N}$, we have: $x_n \in \overline{B}(x_0, \rho_0)$, $d(x_n, x_{n+1}) \in J$, and

$$x_{n+1} \in T(x_n) \cap D, \quad \text{and} \quad d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1)).$$

Furthermore, for each $n, k \in \mathbb{N}$, we have:

$$\begin{aligned} d(x_{n+k}, x_n) &\leq d(x_{n+k}, x_{n+k-1}) + d(x_{n+k-1}, x_{n+k-2}) + \cdots + d(x_{n+1}, x_n) \\ &\leq \psi^{n+k-1}(d(x_0, x_1)) + \cdots + \psi^n(d(x_0, x_1)) \\ &\leq \sum_{i=n}^{n+k-1} \psi^j(d(x_0, x_1)). \end{aligned}$$

Since $\sum_{j=0}^{\infty} \psi^j(d(x_0, x_1)) < \infty$, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $k, n \in \mathbb{N}$ with $n \geq N$, $\sum_{i=n}^{n+k-1} \psi^j$ $(d(x_0, x_1)) < \varepsilon$. Hence, $d(x_{n+k}, x_n) < \varepsilon$ for all $k \in \mathbb{N}$ and $n \geq N$. So, $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in X. Since X is a complete metric space, there exists $\bar{x} \in X$ such that $x_n \to \bar{x}$ as $n \to \infty$. Clearly, $\bar{x} \in D \cap \overline{B}(x_0, \rho_0)$. Since $d(x_n, T(x_n) \cap D) \leq d(x_n, x_{n+1})$ for each $n \in \mathbb{N}$ and, $d(x_n, T(x_n)) \to 0$ as $n \to \infty$. Finally, we get the proof of Theorem 4.1 by following the final proof of Theorem 3.1. □

Next, we give the following slightly modified definition and lemmas from [20–22].

Definition 4.2 ([20]). Let $\varphi : [0, \infty) \to [0, \infty)$. Then φ is said to satisfy the condition (Ψ) if

$(\Psi 1)$ $\varphi(t) < t$ for each $t \in (0, \infty)$;
$(\Psi 2)$ φ is upper semicontinuous from the right on $(0, \infty)$; and
$(\Psi 3)$ there exists a positive real number s such that φ is nondecreasing on $[0, s]$ and $\sum_{n=0}^{\infty} \varphi^n(t) < \infty$ for all $t \in [0, s]$.

Lemma 4.1 ([22]). *If φ satisfies condition (Ψ), then it is a Bianchini–Grandolfi gauge function on $J = [0, s]$.*

Lemma 4.2 ([21]). *Let $\varphi : [0, \infty) \to [0, 1)$ be a function which satisfies*

$$\limsup_{s\to t^+} \varphi(s) < 1 \quad \text{for each } t \in [0, \infty).$$

Then there exists a function $\psi : [0, \infty) \to [0, \infty)$ such that it satisfies condition (Ψ) and $\varphi(t) \cdot t \leq \psi(t)$ for each $t \in [0, \infty)$. Furthermore, from $(\Psi 3)$, $\varphi(t) \cdot t < \psi(t)$ for each $t \in (0, s]$.

Lemma 4.3. *Let X be a metric space, and let $T : X \to \mathcal{CB}(X)$ be a map. Then $d(y, T(y)) \leq d(y, x) + d(x, Tx) + H(T(x), T(y))$ for every $x, y \in X$.*

Remark 4.1. Theorem 4.1 implies that Theorem 1.2. Hence, Theorem 4.1 is a generalization of Nadler's fixed point theorem.

Proof. Suppose that all conditions of Theorem 1.2 hold. Let $D := X$. Since $\limsup_{r\to t^+} \varphi(r) < 1$ for each $t \in [0, \infty)$, by Lemma 4.2, there exists a function ψ such that

(a) $\psi(t) < t$ for each $t \in (0, \infty)$;
(b) ψ is upper semicontinuous from the right on $(0, \infty)$; and
(c) there exists a positive real number s such that ψ is nondecreasing on $[0, s]$ and $\sum_{n=0}^{\infty} \psi^n(t) < \infty$ for all $t \in [0, s]$,
(d) $\varphi(t) \cdot t \leq \psi(t)$ for each $t \in [0, \infty)$; $\varphi(t) \cdot t < \psi(t)$ for each $t \in (0, s]$.

Let $J = [0, s]$. By Lemma 4.1, ψ is a Bianchini–Grandolfi gauge function on $J = [0, s]$.

Now, for each $x \in X$, and each $y \in T(x)$ with $d(x, y) \in J$,

$$d(y, T(y)) \leq H(T(x), T(y)) \leq \varphi(d(x, y)) \cdot d(x, y) \leq \psi(d(x, y)).$$

And for each $x \in X$, and each $y \in T(x)$ with $d(x, y) \in J$ and $x \neq y$,

$$d(y, T(y)) \leq H(T(x), T(y)) \leq \varphi(d(x, y)) \cdot d(x, y) < \psi(d(x, y)).$$

Next, we show that f, defined as $f(x) := d(x, T(x))$, is lower semicontinuous. Indeed, if $\lim_{n\to\infty} x_n = x$, then

$$0 \leq \limsup_{n\to\infty} H(T(x_n), T(x)) \leq \limsup_{n\to\infty} d(x_n, x) = 0.$$

Hence, $\lim_{n\to\infty} H(T(x_n), T(x)) = 0$. Besides, by Lemma 4.3,

$$f(x) \leq d(x_n, x) + f(x_n) + H(T(x_n), T(x)) \quad \text{for each } n \in \mathbb{N}.$$

And this implies that $f(x) \leq \liminf_{n\to\infty} f(x_n)$. Hence, f is lower semicontinuous.

By assumptions, for each $x_0 \in X$, we can construct a sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ such that $x_{n+1} \in T(x_n)$ for all $n \in \mathbb{N} \cup \{0\}$ and $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Hence, there exists $x, y \in X$ such that $y \in T(x)$ and $d(x, y) \in J$. Hence, by Theorem 4.1, there exists $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$. Therefore, Theorem 4.1 implies that Theorem 1.2. □

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